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TWO APPROXIMATION TECHNIQUES FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract—Some authors proposed to approximate the solutions of delay-differential equations by ordinary differential equations to eliminate in part the difficulties presented by time-delays. This technique, termed the chain method, and its modified version are discussed in this paper for general nonlinear time dependent delay differential equations. The modified version is based on some heuristic approximation in biological modelling and the difference in properties of the original and the new modified approximation techniques is demonstrated on a delay differential equation of an epidemic model.

1. INTRODUCTION

One possible way of investigation, as well as numerical solutions of a retarded or functional differential equation, is to approximate the solutions by the solutions of an appropriately chosen system of ordinary differential equations or a sequence of such systems. This method is useful because, among others, the infinite dimensional phase space of functional differential equations is replaced by a finite dimensional space, or the numerical solutions of the original equation is reduced to the numerical solution of ordinary differential equations.

Investigating stability of delay differential equations, in 1965 Repin [1] gave a method ensuring the arbitrarily close approximation of any sufficiently smooth solution of a delay differential equation by corresponding solutions of associated ordinary differential equations. The method avoids the difficulties arising from the gap between infinite and finite dimensional phase spaces by increasing the precision of approximation through increasing the number of ordinary differential equations in the approximating system. The technique termed the *chain method* has been proved by Janushevskij [2] to approximate the characteristic quasipolynomial of linear autonomous retarded equations by the characteristic polynomials of the approximating ordinary equations. So he could use this method to extend some known results for control systems described by ordinary differential equations to control systems with time lags, both in the time and frequency domain.

The applicability of the chain method in numerical computations and in the modelling of biologic processes was demonstrated by Banks [3]. Based on the paper of Banks, an der Heiden [4] pointed out an interesting connection between a hemokinetic and an enzyme synthesis model, showing that the latter can be considered as the chain method approximation of the former.

Combining the basic idea of the chain method with spline theory Banks and Kappel [5] gave a more general approximation technique. Their scheme reduces to the chain method when choosing 0th order spline (i.e. step functions).

Based on some heuristic approximations in biologic compartmental modelling (in particular, that in the paper [6]), this author and his co-workers [7, 8] gave another kind of generalization of the chain method. This generalization makes possible, in particular, to establish a connection between the usual nondelay compartmental systems [9] and the so-called compartmental systems with pipes, described by delay differential equations (see for example Refs [10–13]).

It was shown in papers [7, 8] that for linear autonomous equations with delays the new approximation technique yields the same approximation as the chain method of Repin, so the results obtained in Refs [2, 3, 5] ensure the convergence of the new method in that case.

In the nonlinear case, however, there is no analytic relation between the two approximation techniques, and the fact of convergence is supported only by some computer simulation results.

In this paper, starting with a basic proposition from a previous paper about the approximation of nonlinear partial differential equations by the method of lines [14], the author gives convergence theorems both for the chain method of Repin and for the new approximation technique.

The proofs of the obtained convergence results can be considered elementary as they rely only on the commonly used Gronwall inequality and logarithmic norms of matrices. A new point compared to the results in the papers [1–3, 5] is that here in constructing the approximating equations, time-dependent delays are taken into account.

The approximation method introduced in connection with delay compartmental systems will be presented only in the case of scalar equations important in population and epidemic models and, at the same time, a new convergence theorem will also be proven. The difference in the properties of the two approximation techniques will be demonstrated by investigating the equilibrium states of the original and the approximating differential equations.

2. SOME NOTATIONS, DEFINITIONS AND PROPOSITIONS

In this paper, R^n denotes the space of n-vectors and for any $\mathbf{x} \in R^n$, $|\mathbf{x}|$ denotes the norm of $\mathbf{x}$. The set of n by n matrices is denoted by $R^{n \times n}$ and the norm of $A = (a_{ij}) \in R^{n \times n}$ is denoted by $|A|$.

Let t_0, T be given such that $-\infty < t_0 < T < \infty$ and let γ be a given continuous function on $[t_0, T)$ with the following properties:

$$\gamma(t) > 0, \quad (t_0 \leqslant t < T)$$

and $t - \gamma(t)$ is monotone nondecreasing for $t \in [t_0, T)$.

For any function $\mathbf{x}: [t_0 - \gamma(t_0), T) \to R^n$ and $t \geqslant t_0$, we shall let $\mathbf{x}_t$ denote a translation of the restriction of $\mathbf{x}$ to the interval $[t - \gamma(t), t]$; more specifically, $\mathbf{x}_t: [-\gamma(t), 0] \to R^n$ is defined by

$$\mathbf{x}_t(s) = \mathbf{x}(t + s), \quad -\gamma(t) \leqslant s \leqslant 0.$$

Let us denote by $C([a, b], R^n)$ the space of continuous functions $\phi: [a, b] \to R^n$ with the norm

$$\| \phi \|_{[a, b]} = \max_{a \leqslant s \leqslant b} |\phi(s)|$$

and let $C_t = C([-\gamma(t), 0], R^n)$.

$L([a, b], R^n)$ denotes the Banach space of those functions which are integrable on $[a, b]$ and $L_t = L([-\gamma(t), 0], R^n)$.

If $\mathbf{F}(t, \cdot): D_t \to R^n$ is a given function for any fixed $t \in [t_0, T)$, where D_t is a subset of the collection of functions $\phi: [-\gamma(t), 0] \to R^n$, we say that the relation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}_t) \tag{1}$$

is a retarded functional differential equation.

Definition 2.1

We say that $\mathbf{x}: [t_0 - \gamma(t_0), T) \to R^n$ is a solution of equation (1) if $\mathbf{x}$ is continuous on $[t_0 - \gamma(t_0), T)$, continuously differentiable on $[t_0, T)$ and it satisfies equation (1) on the latter interval;

$$\mathbf{x}: [t_0 - \gamma(t_0), T) \to R^n$$

is a Carathéodory type solution of equation (1) if $\mathbf{x}$ is Lebesgue integrable on $[t_0 - \gamma(t_0), t_0]$, absolutely continuous on $[t_0, T)$ and it satisfies equation (1) almost everywhere on the latter interval;

$$\mathbf{x}: [t_0 - \gamma(t_0), T) \to R^n$$

is a smooth solution of equation (1) if $\mathbf{x}$ is a solution of equation (1) and it is continuously differentiable on $[t_0 - \gamma(t_0), T)$.

Remark 2.1

Any solution of equation (1) is a Carathéodory type solution, too. Moreover, we say that $\mathbf{x} = \mathbf{x}(t_0, \psi)$ is a solution (Carathéodory type solution) of equation (1) belonging to initial function $\phi:[t_0 - \gamma(t_0), R^n)$, if $\mathbf{x}(t_0, \phi)$ is a solution (Carathéodory type solution) of equation (1) and $\mathbf{x}_{t_0}(t_0, \phi) = \phi$.

Remark 2.2

It is very easy to see that any Carathéodory solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ ($\boldsymbol{\phi} \in L_{t_0}$ is a given initial function) is a solution of the following:

$$\left.\begin{aligned} \mathbf{x}(t) &= \phi(0) + \int_{t_0}^{t_0} \mathbf{F}(s, \mathbf{x}_s)\,\mathrm{d}s, \quad t_0 \leqslant t < T, \\ \mathbf{x}_{t_0} &= \phi, \end{aligned}\right\} \tag{2}$$

and conversely, any solution of equations (2) is a Carathéodory solution of equation (1).

Now, we give two existence and uniqueness results which can be proved with some obvious modifications in the proofs of some similar results in Refs [15–17].

Proposition 2.1

Assume that the family of functions $\mathbf{F}(t, \cdot): C_t \rightarrow R^n$, $(t_0 \leqslant t < T)$, satisfies the following conditions:

(i) for any $\mathbf{x} \in C([t_0 - \gamma(t_0), T), R^n)$, $\mathbf{F}(t, \mathbf{x}_t)$ is a continuous function on $[t_0, T)$;

(ii) for any $(t, \mathbf{x})$, $(t, \mathbf{y}) \in [t_0, T) \times C([t_0 - \gamma(t_0), T), R^n)$,

$$|\mathbf{F}(t, \mathbf{y}_t) - \mathbf{F}(t, \mathbf{x}_t)| \leqslant m(t) \,\|\mathbf{y}_t - \mathbf{x}_t\|_{[\gamma(t), 0]}, \quad t_0 \leqslant t < T, \tag{3}$$

where $m(t) \geqslant 0$ is a continuous function.

Then for any $\boldsymbol{\phi} \in C_{t_0}$, equation (1) has exactly one solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ belonging to $\boldsymbol{\phi}$ and for any $\phi \in C_{t_0}$:

$$|\mathbf{x}(t_0, \phi)(t) - \mathbf{x}(t_0, \psi)(t)| \leqslant \|\Phi - \psi\|_{[-\gamma(t), 0]} \exp\left[\int_{t_0}^{t} m(s)\,\mathrm{d}s\right], \quad t_0 \leqslant t < T \tag{4}$$

Proof. The proof of this proposition can be done by application of Schauder's fixed point theorem and Gronwall's lemma, similar to the proof of analogous theorems in Refs [16, 17].

Proposition 2.2

Assume that the functional family $\mathbf{F}(t, \cdot): L_t \rightarrow R^n (t_0 \leqslant t < T)$, satisfies the conditions:

(i) for any locally integrable function $\mathbf{x}:[t_0 - \gamma(t_0), T) \rightarrow R^n$, $\mathbf{F}(t, \mathbf{x}_t)$ is locally integrable on $[t_0, T)$;

(ii) for any locally integrable $\mathbf{x}, \mathbf{y}:[t_0 - \gamma(t_0), T) \rightarrow R^n$,

$$|\mathbf{F}(t, \mathbf{x}_t) - \mathbf{F}(t, \mathbf{y}_t)| \leqslant m(t) \left\{ \sum_{i=0}^{N} |\mathbf{x}_t(-\gamma_i(t)) - \mathbf{y}_t(-\gamma_i(t))| + \int_{-\gamma(t)}^{0} |\mathbf{x}_t(s) - \mathbf{y}_t(s)|\,\mathrm{d}s \right\}, \tag{5}$$

where $m(t) \geqslant 0$ is a continuous function, $\gamma_i(t)$ is differentiable on $[t_0, T)$ moreover $0 \leqslant \gamma_i(t) \leqslant \gamma(t)$,

$$\gamma_i(t) \leqslant 1 - \epsilon, \quad \gamma t_0 \leqslant t < T, \quad i = \overline{0, N}), \quad \text{and} \quad \epsilon \in (0, 1).$$

Then, for any $\boldsymbol{\phi} \in L_{t_0}$ equation (1) has exactly one Carathéodory solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ belonging to $\boldsymbol{\phi}$ and for any $T_1 \in [t_0, T)$,

$$\|\mathbf{x}(t_0, \boldsymbol{\phi}) - \mathbf{x}(t_0, \boldsymbol{\psi})\|_{[t_0, T_1]} \rightarrow 0, \tag{6}$$

if $\boldsymbol{\phi}, \boldsymbol{\psi} \in L_{t_0}$ are such that

$$|\boldsymbol{\phi}(0) - \boldsymbol{\psi}(0)| + \int_{-\gamma(t_0)}^{0} |\boldsymbol{\phi}(s) - \boldsymbol{\psi}(s)|\,\mathrm{d}s \to 0. \tag{7}$$

Moreover,

$$\| \mathbf{x}(t, \boldsymbol{\phi}) - \mathbf{x}(t_0, \boldsymbol{\psi}) \|_{[t_0, T)} \to 0. \tag{8}$$

if $\boldsymbol{\phi}, \boldsymbol{\psi} \in L_{t_0}$ are such that condition (7) holds and

$$\gamma_M = \sup_{t_0 \leqslant t < T} \gamma(t) < \infty, \quad \int_{t_0}^{T} m(t)\,\mathrm{d}t < \infty. \tag{9}$$

Proof. The existence and uniqueness of the solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ is a consequence of Section 2.6 from Ref. [15].

Now, let us consider $t_0 < T_1 < T$,

$$\gamma_1 = \sup_{t_0 \leqslant t \leqslant T_1} \gamma(t), \quad \tau_i(t) = t - \gamma_i(t), \quad t_0 \leqslant t < T,$$

and let $z:[t_0 - \gamma_1, T_1] \to R_+$ be a locally integrable function. Then with some quite elementary manipulations, we have:

$$\begin{aligned}\int_{t_0}^{t} m(u) z(\tau_i(u))\,\mathrm{d}u &= \int_{\tau_i^{-1}(t_0)}^{\tau_i^{-1}(t)} \frac{1}{\dot{\tau}_i(\tau_i^{-1}(v))} m(\tau_i^{-1}(v)) z(v)\,\mathrm{d}v \\ &\leqslant c_i \int_{-\gamma_i(t_0)}^{0} z(t_0 + u)\,\mathrm{d}u + \int_{t_0}^{t} m_i(u) z(u)\,\mathrm{d}u, \quad i = \overline{0, N},\end{aligned} \tag{10}$$

and

$$\begin{aligned}\int_{t_0}^{t} \left\{ m(\tau) \int_{-\gamma_1}^{0} z(\tau + s)\,\mathrm{d}s \right\} \mathrm{d}\tau &= \int_{-\gamma_1}^{0} \left(\int_{t_0 + s}^{t + s} m(u - s) z(u)\,\mathrm{d}u \right) \mathrm{d}s \\ &\leqslant \int_{-\gamma_1}^{0} \left(\int_{t + s}^{t_0} m(u - s) z(u)\,\mathrm{d}u \right) \mathrm{d}s + \int_{-\gamma_1}^{0} \left(\int_{t_0}^{t} m(u - s) z(u)\,\mathrm{d}u \right) \mathrm{d}s \\ &\leqslant C_{N+1} \int_{-\gamma_1}^{0} z(t_0 + u)\,\mathrm{d}u + \int_{t_0}^{t} m_{N+1}(u) z(u)\,\mathrm{d}u, \quad t_0 \leqslant t < T_1,\end{aligned} \tag{11}$$

where for $i = \overline{0, N}$,

$$C_i = \sup_{t_0 - \gamma(t_0) \leqslant u \leqslant t_0} m_i(u), \quad m_i(u) = \frac{1}{\dot{\tau}_i(\tau_i^{-1}(u))} m(\tau_i^{-1}(u)), \quad t_0 \leqslant u < T,$$

and

$$C_{N+1} = m_{N+1}(t_0), \quad m_{N+1}(u) = \int_{\mu}^{u + \gamma_1} m(s)\,\mathrm{d}s, \quad t_0 \leqslant u < T.$$

Now we consider two arbitrary solutions $\mathbf{x} = \mathbf{x}(t_0, \boldsymbol{\phi})$ and $\mathbf{y} = \mathbf{y}(t_0, \boldsymbol{\psi})$, $(\boldsymbol{\phi}, \boldsymbol{\psi} \in L_{t_0})$, and we extend these solutions for $[t_0 - \gamma_1, t_0 - \gamma(t_0)]$ so that

$$\mathbf{x}(t) = \mathbf{y}(t) = \boldsymbol{\theta}, \quad t_0 - \gamma_1 \leqslant t \leqslant t_0 - \gamma(t_0).$$

Then from condition (5), we have

$$|\mathbf{F}(t, \mathbf{x}_t) - \mathbf{F}(t, \mathbf{y}_t)| \leqslant m(t) \left\{ \sum_{i=0}^{N} |\mathbf{x}_t(-\gamma_i(t)) - \mathbf{y}_t(-\gamma_i(t))| + \int_{-\gamma_1}^{0} |\mathbf{x}_t(s) - \mathbf{y}_t(s)|\,\mathrm{d}s, \right.$$

and thus equation (2), which is equivalent to equation (1), yields that

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leqslant |\boldsymbol{\phi}(0) - \boldsymbol{\psi}(0)| + \int_{t_0}^{t} m(u) \left\{ \sum_{i=0}^{N} |\mathbf{x}_u(-\gamma_i(u)) - \mathbf{y}_u(-\gamma_i(u))| + \int_{-\gamma_1}^{0} |\mathbf{x}_u(s) - \mathbf{y}_u(s)|\mathrm{d}s \right\} \mathrm{d}u.$$

Using equations (10) and (11), we have

$$|\mathbf{x}(t)-\mathbf{y}(t)| \leqslant k_{\phi,\psi} + \int_{t_0}^{t} \sum_{i=0}^{N+1} m_i(u)|\mathbf{x}(u)-\mathbf{y}(u)|\,du, \quad t_0 \leqslant t \leqslant T,$$

where

$$k_{\phi,\psi} = |\boldsymbol{\phi}(0)-\boldsymbol{\psi}(0)| + \sum_{i=0}^{N} C_i \int_{-\gamma_i(t_0)}^{0} |\mathbf{x}_{t_0}(u) - y_{t_0}(u)|\,du + C_{N+1}\int_{-\gamma_1}^{0} |\mathbf{x}_{t_0}(u)-\mathbf{y}_{t_0}(u)|\,du$$

$$\leqslant \max\left\{1, \sum_{i=0}^{N+1} C_i\right\}\left[|\boldsymbol{\phi}(0)-\boldsymbol{\psi}(0)| + \int_{-\gamma(t_0)}^{0} |\boldsymbol{\phi}(u)-\boldsymbol{\psi}(u)|\,du\right].$$

From this, by Gronwall's inequality, we obtain

$$|\mathbf{x}(t)-\mathbf{y}(t)| \leqslant k_{\phi,\psi}\exp\left(\int_{t_0}^{t}\sum_{i=0}^{N+1} m_i(u)\,du\right), \quad \text{on} \quad [t_0, T_1]. \tag{12}$$

However, this inequality is valid on $[t_0, T)$ if equation (9) holds; moreover, in this case

$$\int_{t_0}^{T} m_i(u)\,du < \infty, \quad i = \overline{0, N+1}.$$

Thus, conditions (6) and (8) are evident corollaries of inequality (12). Now we prove the following useful lemma.

Lemma 2.1

Let

$$\mathbf{G}: C([-r,0], R^n) \to R^n, \quad 0 < r < \infty,$$

be a continuous function and

$$|\mathbf{G}(\boldsymbol{\phi}) - \mathbf{G}(\boldsymbol{\psi})| \leqslant K\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{[-r,0]}, \tag{13}$$

for any $\boldsymbol{\phi}, \boldsymbol{\psi} \in C([-r,0], R^n)$, where $K > 0$ is a constant. Then the set

$$S_G = \{\boldsymbol{\phi} \in C^1([-r,0], R^n) : \dot{\boldsymbol{\phi}}(0-) = \mathbf{G}(\boldsymbol{\phi})\}$$

is dense in $C([-r,0], R^n)$.

Proof. The proof will be complete if we prove that S_G is dense in $C^1([-r,0], R^n)$ since this set is dense in $C([-r,0], R^n)$.

Let $\boldsymbol{\phi} \in C^1([-r,0], R^n)$ be an arbitrary fixed function and $\delta \in (0, r)$ an arbitrary real number, moreover let us define $\mathbf{z}^{(\delta)}$ by

$$\mathbf{z}^{(\delta)}(t) = \begin{cases} \dfrac{1}{\delta}[\mathbf{G}(\boldsymbol{\phi}) - \dot{\boldsymbol{\phi}}(0-)](t_0+\delta-t), & t_0 \leqslant t \leqslant t_0+\delta \\ 0, & t_0+\delta < t \leqslant t_0 + r. \end{cases} \tag{14}$$

Then, using condition (13), we get that the equation

$$\begin{cases} \dot{\mathbf{x}}^{(\delta)}(t) = \mathbf{G}(\mathbf{x}^{(\delta)}_{[t]}) + \mathbf{z}^{(\delta)}(t), & t_0 \leqslant t \leqslant t_0 + r, \\ \mathbf{x}^{(\delta)}_{[t_0]} = \boldsymbol{\phi}, \end{cases} \tag{15}$$

has exactly one solution $\mathbf{x}^{(\delta)}(t_0, \boldsymbol{\phi})$ on $[t_0 - r, t_0 + r)$, where $\mathbf{x}^{(\delta)}_{[t]}$ is defined by $\mathbf{x}^{(\delta)}_{[t]}(s) = \mathbf{x}^{(\delta)}(t+s)$, $-r \leqslant s \leqslant 0$.

From equation (15), we get

$$|\mathbf{x}^{(\delta)}_{[t]}(s) - \mathbf{x}^{(\delta)}_{[t_0]}(s)| \leqslant |\mathbf{x}^{(\delta)}_{[t_0]}(s) - \mathbf{x}^{(\delta)}(t_0)| + \int_{t_0}^{t} |\mathbf{G}(\mathbf{x}^{(\delta)}_{[t_0]}) - \mathbf{z}^{(\delta)}(u)|\,du$$

$$+ \int_{t_0}^{t} |\mathbf{G}(\mathbf{x}^{(\delta)}_{[u]}) - \mathbf{G}(\mathbf{x}^{(\delta)}_{[t_0]})|\,du, \quad t_0 \leqslant t+s \leqslant t_0 + r.$$

This and inequality (13) yields

$$|\mathbf{x}_{[t]}^{(\delta)}(s) - \mathbf{x}_{[t_0]}^{(\delta)}(s)| \leqslant k_\phi + K \int_{t_0}^{t} \| x_{[u]}^{(\delta)} - \mathbf{x}_{[t_0]}^{(\delta)} \|_{[-r,0]} \,\mathrm{d}u,$$

thus, since $\mathbf{x}_{[t_0]}^{(\delta)}(s) = \boldsymbol{\phi}(s)$, we have

$$\|\mathbf{x}_{[t]}^{(\delta)} - \boldsymbol{\phi}\|_{[-r,0]} \leqslant k_\phi + K \int_{t_0}^{t} \|\mathbf{x}_{[u]}^{(\delta)} - \boldsymbol{\phi}\|_{[-r,0]} \,\mathrm{d}u, \quad t_0 \leqslant t \leqslant t_0 + r,$$

where

$$k_\phi = \|\boldsymbol{\phi} - \boldsymbol{\phi}(0)\|_{[-r,0]} + r(|\mathbf{G}(\boldsymbol{\phi})| + |\mathbf{G}(\boldsymbol{\phi}) - \dot{\boldsymbol{\phi}}(0-)|).$$

From the last inequality, by Gronwall's lemma, we have

$$\|\mathbf{x}_{[t]}^{(\delta)} - \boldsymbol{\phi}\|_{[-r,0]} \leqslant k_\phi \exp(K(t - t_0)), \quad t_0 \leqslant t \leqslant t_0 + r$$

and thus

$$\sup_{0 \leqslant \delta \leqslant r} \|\mathbf{x}_t^{(\delta)}\|_{[-r,0]} < \infty. \tag{16}$$

Let us define

$$\boldsymbol{\psi}^{(\delta)}(u) = \mathbf{x}_{[t_0+\delta]}^{(\delta)}(u) = \mathbf{x}^{(\delta)}(t_0 + \delta + u), \quad -r \leqslant u \leqslant 0.$$

Then $\boldsymbol{\psi}^{(\delta)}(u)$ is continuously differentiable on $-r \leqslant u \leqslant -\delta$ and on this interval $\dot{\boldsymbol{\psi}}^{(\delta)}(u) = \dot{\boldsymbol{\phi}}(\delta + u)$. On the other hand, $\boldsymbol{\psi}^{(\delta)}(u)$ is continuously differentiable on $-\delta < u < 0$ and

$$\dot{\boldsymbol{\psi}}^{(\delta)}(u) = \dot{\mathbf{x}}^{(\delta)}(t_0 + \delta + u) = \mathbf{G}(\mathbf{x}_{[t_0+\delta+u]}) + \mathbf{z}^{(\delta)}(t_0 + \delta + u). \tag{17}$$

From these, using equation (14), we obtain

$$\dot{\boldsymbol{\psi}}^{(\delta)}(-\delta+) = \mathbf{G}(\mathbf{x}_{[t_0]}) + \mathbf{z}^{(\delta)}(t_0) = \dot{\boldsymbol{\phi}}(0-) = \dot{\boldsymbol{\psi}}^{(\delta)}(-\delta-).$$

This implies that $\boldsymbol{\psi}^{(\delta)}(u)$ is a continuously differentiable function on $[-r, 0]$ and from equation (17),

$$\dot{\boldsymbol{\psi}}^{(\delta)}(0-) = \mathbf{G}(\mathbf{x}_{[t_0+\delta]}) + \mathbf{z}^{(\delta)}(t_0 + \delta) = \mathbf{G}(\boldsymbol{\psi}^{(\delta)}),$$

that is, for any $\delta \in (0, r)$ the function $\boldsymbol{\psi}$ belongs to S_G. Equations (15) and (16) yield

$$m = \sup_{0 \leqslant \delta \leqslant r} \max_{t_0 - r \leqslant u \leqslant t_0 + r} |\dot{x}^{(\delta)}(u)| < \infty,$$

thus

$$\|\boldsymbol{\psi}^{(\delta)} - \boldsymbol{\phi}\| = \|\mathbf{x}_{[t_0+\delta]}^{(\delta)} - \mathbf{x}_{[t_0]}^{(\delta)}\|_{[-r,0]} \leqslant m \cdot \delta \to 0, \quad \text{as} \quad \delta \to 0. \tag{18}$$

This means that for the arbitrarily fixed $\boldsymbol{\phi} \in C^1([-r, 0], R^n)$ there exists a function series $\boldsymbol{\psi}^{(\delta)} \in S_G$ such that condition (18) holds. Thus, S_G is dense in $C^1([-r, 0], R^n)$.

Remark 2.3

The statement of Lemma 2.1 remains valid if we replace the condition (13) by the following weaker one: for any $\boldsymbol{\phi} \in C([-r, 0], R^n)$ the initial value problem

$$\dot{\mathbf{x}}^{(0)}(t) = \mathbf{G}(\mathbf{x}_{[t]}^{(0)}), \quad t_0 \leqslant t \leqslant t_0 + r_0,$$

$$\mathbf{x}_{[t_0]}^{(0)} = \phi,$$

has exactly one solution on $[t_0 - r, t_0 + r_0]$, where $r_0 > 0$ is a fixed real number. In this case we cannot use Gronwall's inequality. But instead of this, using Schauder's fixed point theorem in the same way as in Lemma 2.3 in Ref. [15], we get that for any small enough $\delta > 0$ the solution $\mathbf{x}^{(\delta)}$ of equation (15) exists and

$$\sup_{0 \leqslant \delta \leqslant \delta_0} \|\mathbf{x}^{(\delta)} - \mathbf{x}_{[t]}^{(0)}\|_{[-r,0]} < \infty.$$

After this, all steps of the proof of Lemma 2.1 apply without change.

Now, using Lemma 2.1 we prove the most important results of this section.

Proposition 2.3

If conditions (i) and (ii) of Proposition 2.1 are satisfied, then for any solution $\mathbf{x}$ of equation (1) there exists a sequence of smooth solutions $\mathbf{x}^{(k)}$, $(k \geqslant 1)$, of equation (1) such that

$$\sup_{t_0 - \gamma(t_0) \leqslant t \leqslant T_1} |\mathbf{x}(t) - \mathbf{x}^{(k)}(t)| \to 0, \quad \text{as} \quad k \to +\infty, \tag{19}$$

for any $T_1 \in [t_0, T)$. If, in addition to conditions (i) and (ii) in Proposition 2.1, we suppose

$$\int_{t_0}^{\infty} m(t)\, \mathrm{d}t < \infty, \tag{20}$$

then

$$\sup_{t_0 - \gamma(t_0) \leqslant t < T} |\mathbf{x}(t) - \mathbf{x}^{(k)}(t)| \to 0, \quad k \to +\infty. \tag{21}$$

Proof. Let $\mathbf{x}$ be an arbitrary fixed solution on $[t_0 - \gamma(t_0), T)$ and $\boldsymbol{\phi} = \mathbf{x}_{t_0}$. Then by Lemma 2.1, we have the existence of a function sequence $\{\boldsymbol{\phi}^{(k)}\}_{k=1}^{\infty}$ such that $\boldsymbol{\phi}^{(k)} \in S_G$, where

$$\mathbf{G}(\cdot) = \mathbf{F}(t_0, \cdot) \quad \text{and} \quad \|\boldsymbol{\phi}^{(k)} - \boldsymbol{\phi}\|_{[-\gamma(t_0), 0]} \to 0, \quad k \to +\infty.$$

On the other hand, the solution $\mathbf{x}^{(k)} = \mathbf{x}^{(k)}(t_0, \boldsymbol{\phi}^{(k)})$ of equation (1) exists on $[t_0, T)$ and from inequality (4), we have

$$|\mathbf{x}(t) - \mathbf{x}^{(k)}(t)| \leqslant \|\boldsymbol{\phi}^{(k)} - \boldsymbol{\phi}\|_{[-\gamma(t_0), 0]} \exp\left(\int_{t_0}^{t} m(s)\, \mathrm{d}s \right), \quad t_0 < t < T.$$

However, this means that condition (19) and, under condition (20), condition (21) is valid. Thus, it remains to show that $\mathbf{x}^{(k)} = \mathbf{x}^{(k)}(t_0, \boldsymbol{\phi}^{(k)})$, $k \geqslant 1$, are smooth solutions. This is evident because

$$\dot{\boldsymbol{\phi}}^{(k)}(0-) = \mathbf{F}(t_0, \boldsymbol{\phi}^{(k)}), \quad k \geqslant 1,$$

and thus

$$\mathbf{x}^{(k)}(t_0, \boldsymbol{\phi}^{(k)})(t_0+) = F(t_0, \boldsymbol{\phi}^{(k)}) = \mathbf{x}^{(k)}(t_0, \boldsymbol{\phi}^{(k)})(t_0-), \quad k \geqslant 1.$$

Proposition 2.4

Assume that conditions (i) and (ii) of Proposition 2.2 are valid and, in addition, the function $F(t, \mathbf{x}_t)$ is continuous for any $\mathbf{x} \in C([t_0 - \gamma(t_0), T], R^n)$. Then for any Carathéodory solution $\mathbf{x}$ of equation (1) there exists a *sequence* of smooth solutions $\mathbf{x}^{(k)}$, $(k \geqslant 1)$, of equation (1) such that

$$|\mathbf{x}(t) - \mathbf{x}^{(k)}(t)| \to 0, \quad \text{as} \quad k \to +\infty, \tag{22}$$

uniformly on any interval $[t_0, T_1] \subset [t_0, T)$ and

$$|\mathbf{x}(t_0) - \mathbf{x}^{(k)}(t_0)| + \int_{-\gamma(t_0)}^{0} |\mathbf{x}_{t_0}(u) - \mathbf{x}_{t_0}^{(k)}(u)| \mathrm{d}u \to 0, \quad \text{as} \quad k \to \infty. \tag{23}$$

Moreover, if in addition to conditions (i) and (ii) in Proposition 2.2, we suppose condition (20) then the convergence (22) is uniform on $[t_0, T)$.

Proof. The statements of the present proposition for any solution $\mathbf{x}$ of equation (1) belonging to a continuous initial function are valid, since from condition (5) the estimate (3) follows.

If $\mathbf{x}$ is an arbitrarily fixed Carathéodory solution then there is a sequence $\{\boldsymbol{\psi}_k\}_{k=1}^{\infty} \subset C_{t_0}$ such that

$$|\mathbf{x}(t_0) - \boldsymbol{\psi}_k(0)| + \int_{-\gamma(t_0)}^{0} |\mathbf{x}_{t_0}(u) - \boldsymbol{\psi}_k(u)| \mathrm{d}u \to 0, \quad k \to +\infty,$$

and the solution

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k)}(t_0, \boldsymbol{\psi}_k), \quad (k \geqslant 1),$$

of equation (1) exists on $[t_0 - \gamma(t_0), T)$.

However, in this case conditions (6) and (8) imply the statements of the present proposition.

3. APPROXIMATION THEOREMS

In this section, for any $\mathbf{x} \in R^n$ and $A \in R^{n \times n}$, we use the following special norms:

$$|\mathbf{x}|_0 = \max_{1 \leqslant i \leqslant n} |x_i|, \quad |A|_0 = \sup_{1 \leqslant k \leqslant n} \sum_{i=1}^{k} |a_{ik}|$$

and

$$|\mathbf{x}|_1 = \sum_{i=1}^{n} |x_i|, \quad |A|_1 = \sup_{1 \leqslant i \leqslant n} \sum_{k=1}^{n} |a_{ik}|.$$

The importance of these norms is that when using them it is simple to calculate the logarithmic norms of A and to get estimates for solutions of the linear ordinary differential equation $\dot{x} = Ax$ (see for example Ref. [18]).

Let A be a given n by n matrix. Then the logarithmic norm of A, corresponding to the norm $|\cdot|_0$ is

$$\mu_0[A] = \lim_{h \to 0} \frac{|I_n + hA|_0 - 1}{h} = \sup_{1 \leqslant i \leqslant n} \left[a_{ii} + \sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ik}| \right],$$

and that corresponding to $|\cdot|_1$ is

$$\mu_1[A] = \lim_{h \to 0} \frac{|I_n + hA|_1 - 1}{h} = \sup_{1 \leqslant k \leqslant n} \left[a_{kk} + \sum_{\substack{i=1 \\ i \neq k}}^{n} |a_{ik}| \right].$$

Lemma 3.1

Let $l \geqslant 2$ be a given integer and $\delta = 0$ or 1. Assume that $A^{(l,j)}(t)$, $(j = \overline{0, l})$, is an arbitrary n by n continuous matrix function, $D^{(l,j)}(t)$ is a diagonal and continuous n by n matrix function with non-negative elements defined on the interval $[t_0, T)$ $(t_0 \leqslant T \leqslant \infty)$. For any fixed $t \in [t_0, T]$, $A_\delta^{(l)}(t)$ denotes the matrix of the linear transformation $L(t): R^{(l+1)n}\, R^{(l+1)n}$;

$$\mathbf{y}^{(l)}(t) = L(t)\mathbf{x}^{(l)} := A_\delta^{(l)}(t)\mathbf{x}^{(l)}, \quad \mathbf{x}^{(l)} = (\mathbf{x}^{(l,0)}, \ldots, \mathbf{x}^{(l,l)})^{\mathrm{T}} \in R^{(l+1)n},$$

where the vector components $\mathbf{x}^{(l,i)} \in R^n$ and $\mathbf{y}^{(l,i)} \in R^n$ of $\mathbf{x}^{(l)}$ and $\mathbf{y}^{(l)}(t) = (\mathbf{y}^{(l,0)}(t), \ldots, \mathbf{y}^{(l,l)}(t))$ are connected by

$$\left.\begin{aligned} \mathbf{y}^{(l,0)}(t) &= -\delta D^{(l,0)}(t)\mathbf{x}^{(l,0)} + \sum_{j=0}^{l} A^{(l,j)}\mathbf{x}^{(l,j)} + \delta D^{(l,l)}(t)\mathbf{x}^{(l,l)} \\ \mathbf{y}^{(l,i)}(t) &= -D^{(l,i)}(t)\mathbf{x}^{(l,i)} + [\delta D^{(l,i-1)}(t) + (1-\delta)D^{(l,i)}(t)]\mathbf{x}^{(l,i-1)}. \end{aligned}\right\} \tag{24}$$

Then $A_\delta^{(l)}(t)$ is continuous, moreover

$$\mu_0[A_\delta^{(l)}(t)] = \max\left\{0, \mu_0\left[A^{(l,0)}(t) + \sum_{j=1}^{l} \mathrm{abs}(A^{(l,j)}(t))\right]\right\}, \tag{25}$$

where for $A^{(l,j)} = (a_{ik}^{(l,j)})$ we denoted the matrix $(|a_{ik}^{(l,j)}|)$ by abs $(A^{(l,j)})$.

Proof. The proof is quite elementary therefore we do not detail it.

Now, we prove Proposition 3.1.

Proposition 3.1

Let t_0, $\tau_0 \geqslant 0$ real numbers, $t_0 < T_1 \leqslant \infty$ and $\delta = 0$, or 1 and assume:

(i) $\bar{V}$ and $V \subset \bar{V}$ are two subsets from a collection of functions $\mathbf{x}: [t_0 - \tau, T_1] \to R^n$, and the functions

$$\mathbf{a}^{(l,i)}: [t_0, T_1] x \bar{V} \to R^{N_l}, \boldsymbol{\alpha}^{(l,i)}: \bar{V} \to R^{N_l}, \quad N_l = (l+1)n, \quad l \geqslant 2, \quad i = \overline{0, l},$$

are such that for any $\mathbf{x} \in V$ the function $\mathbf{a}^{(l)}(t, \mathbf{x}) = (a^{(l,0)}(t, \mathbf{x}), \ldots, a^{(l,l)}(t, \mathbf{x}))^{\mathrm{T}}$ is differentiable in t, and

$$|\boldsymbol{\alpha}^{(l)}(\mathbf{x}) - \mathbf{a}^{(l)}(t_0, \mathbf{x})|_\delta \to 0, \quad \text{as} \quad l \to +\infty, \tag{26}$$

moreover for any $\mathbf{x} \in \bar{V}$ there exists a function sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subset V$ such that

$$\left.\begin{aligned} &\sup_{l \geqslant 2} \sup_{t_0 \leqslant t \leqslant T_1} |\mathbf{a}^{(l,0)}(t, \mathbf{x}_k) - \mathbf{a}^{(l,0)}(t, \mathbf{x})| \to 0, \quad \text{as} \quad k \to +\infty, \\ &\sup_{l \geqslant 2} |\alpha^{(l)}(\mathbf{x}_k) - \boldsymbol{\alpha}^{(l)}(\mathbf{x})| \to 0, \quad \text{as} \quad k \to +\infty; \end{aligned}\right\} \tag{27}$$

(ii) the n by n matrix functions $A^{(l,j)}(t)$ and $D^{(l,k)}(t)$, $(j = \overline{0, l})$, are continuous on $[t_0, T]$ in matrix norm $|\cdot|_\delta$, $D^{(l,j)}(t)$ is diagonal matrix with non-negative elements, moreover

$$\max\left\{0, \mu_0[A^{(l,0)}(t)] + \sum_{j=1}^{l} \text{abs}[A^{(l,j)}(t)]\right\} \leqslant \lambda_0, \quad t_0 \leqslant t \leqslant T_1, \quad \text{if} \quad \delta = 0 \tag{28}$$

and

$$\mu_1[A_1^{(l)}(t)] \leqslant \lambda_1 \quad t_0 \leqslant t \leqslant T_1, \quad j = 0, l \tag{29}$$

if $\delta = 1$, where $\lambda_\delta (\delta = 0, 1)$, are real numbers;

(iii) $\mathbf{H}^{(l,i)}: [t_0, T_1] x R^{N_l} \to R^n$ is a continuous function and for any $\mathbf{x}, \mathbf{y} \in R^{N_l}$

$$|\mathbf{H}^{(l,i)}(t, \mathbf{x}) - \mathbf{H}^{(l,i)}(t, \mathbf{y})|_\delta \leqslant m(t)|\mathbf{x} - \mathbf{y}|_\delta, \quad i = 0, 1, \quad l \geqslant 2, \tag{30}$$

where $m \in C([t_0, T], R_+)$:

(iv) for any $\mathbf{x} \in V$, there exist functions $\epsilon^{(l)}(\mathbf{x})$ from $C([t_0, T_1], R^n)$, $(l \geqslant 2)$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{a}^{(l)}(t, \mathbf{x}) = A_\delta^{(l)}(t)\mathbf{a}^{(l)}(t, \mathbf{x}) + H^{(l)}(t, \mathbf{a}^{(l)}(t, x)) + \epsilon^{(l)}(\mathbf{x})(t), \tag{31}$$

on $[t_0, T)$, where $A_\delta^{(l)}(t)$ is defined in Lemma 3.1, and

$$\mathbf{H}^{(l)} = (\mathbf{H}^{(l,0)}, \mathbf{H}^{(l,1)}, \boldsymbol{\theta}_n, \ldots, \boldsymbol{\theta}_n),$$

where $\boldsymbol{\theta}_n \in R^n$ denotes the zero vector.

Then

(a) for any $\mathbf{x} \in V$ and $l \geqslant 2$, we have

$$|\mathbf{a}^{(l,0)}(t, \mathbf{x}) - \mathbf{x}^{(l,0)}(t)|_\delta \leqslant \exp(\lambda_\delta t) \int_{t_0}^{t} \exp(-\lambda_\delta s)|\epsilon^{(l)}(\mathbf{x})(s)| \,\mathrm{d}s \times \exp\left[\int_{t_0}^{t} m|s| \,\mathrm{d}s\right], \quad t_0 \leqslant t \leqslant T_1, \tag{32}$$

where $\mathbf{x}^{(l,0)}$ is the 0th vector component of the solution $\mathbf{x}^{(l)} = (\mathbf{x}^{(l,0)}, \ldots, \mathbf{x}^{(l,l)})^{\mathrm{T}}$ of

$$\left.\begin{aligned} &\dot{\mathbf{x}}^{(l)}(t) = A_\delta^{(l)}(t)\mathbf{x}^{(l)}(t) + \mathbf{H}^{(l)}(t, \mathbf{x}^{(l)}(t)), \quad t_0 \leqslant t \leqslant T_1, \\ &\mathbf{x}^{(l)}(t_0) = \alpha^{(l)}(\mathbf{x}), \quad l \geqslant 2; \end{aligned}\right\} \tag{33}$$

(b) if $T_1 < \infty$ and for any $\mathbf{x} \in V$,

$$\max_{t_0 \leqslant t \leqslant T_1} |\epsilon^{(l)}(\mathbf{x})(t)| \to 0, \quad l \to +\infty, \tag{34}$$

then for any $\mathbf{x} \in V$,

$$\sup_{t_0 \leqslant t \leqslant T_1} |\mathbf{a}^{(l,0)}(t, \mathbf{x}) - \mathbf{x}^{(l,0)}(t)| \to 0, \quad l \to +\infty; \tag{35}$$

(c) if $T_1 = +\infty$, $\lambda_\delta \leqslant 0$, $\int_{t_0}^{\infty} m(t)\,\mathrm{d}t < \infty$ and for any $\mathbf{x} \in \bar{V}$

$$\int_{t_0}^{\infty} |\epsilon^{(l)}(\mathbf{x})(\mathrm{t})|_\delta\,\mathrm{d}t \to 0, \quad \text{as} \quad l \to +\infty, \tag{36}$$

then for any $\mathbf{x} \in \bar{V}$

$$\sup_{t_0 \leqslant t < \infty} |\mathbf{a}^{(l,0)}(t, \mathbf{x})| - \mathbf{x}^{(l,0)}(t)|_\delta \to 0, \quad \text{as} \quad l \to +\infty, \tag{37}$$

Proof. The statements of this proposition are an obvious corollary of Proposition 2.2 of Ref. [14] if we use Lemma 3.1 under conditions (i)–(iv) of this proposition and the following definition of $P^{(l)} \in R^{N_l \times N_l}$:

$$P^{(l)} = \begin{pmatrix} I_n & \theta_n \cdots \theta_n \\ \theta_n & \theta_n \cdots \theta_n \end{pmatrix},$$

where I_n and θ_n denote the identity and zero matrix in $R^{n \times n}$, respectively.

Using this general result we are able to give many variations of approximation statements concerning functional differential equaions.

Theorem 3.1

Suppose t_0 is a real number, $t_0 < T < \infty$, and the following conditions hold:

(i) γ and $\gamma_i (i = \overline{1, k})$, are continuous functions on $[t_0, T)$, $\gamma(t) > 0$, $1 - \dot{\gamma}(t) \geqslant 0$ and $0 \leqslant \gamma_i(t) \leqslant \gamma(t)$ for $\mathrm{t} \in [t_0, T)$;

(ii) $A_0 : [t_0, T) \to R^{n \times n}$ and $\mathbf{f} : [t_0, T] \times R^{(k+1)n} \to R^n$ are continuous and

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|_0 \leqslant m(t)|\mathbf{x} - \mathbf{y}|_0, \tag{38}$$

for any $(t, \mathbf{x})$, $(t, \mathbf{y}) \in [t_0, T] x R^{(k+1)n}$, where $m \in C([t_0, T], R_+)$;

(iii) $Q(t, s)$ is an n by n matrix function $Q(t, s) = 0$, $(s \leqslant \gamma(t))$, $Q(t, s) - Q(t, 0)$ is continuous in $t \in [t_0, T]$ for any $s \geqslant 0$ and Q is of bounded variation in s on $[-\gamma(t), 0]$

$$\mathrm{Var}[-\gamma(t), 0]Q(t, \cdot) \leqslant m_1(t), \quad t_0 \leqslant t \leqslant T, \tag{39}$$

where $m_1 \in C([t_0, T), R_+)$, moreover the functions

$$Q\left(t, -j\frac{\gamma(t)}{l}\right), \quad (j = \overline{1, l})$$

and

$$L(t, \mathbf{x}_t) := \int_{-\gamma(t)}^{0} [\mathrm{d}sQ(t, s)]\mathbf{x}(t + s), \quad \mathbf{x} \in C([t_0 - \gamma(t_0), T], R^n), \tag{40}$$

are continuous on $[t_0, T]$.

Then

(a) for any $\boldsymbol{\phi} \in C_{t_0}$ the solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ of

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) = A_0(t)\mathbf{x}(t) + \int_{-\gamma(t)}^{0} [\mathrm{d}sQ(t, s)]\mathbf{x}_t(s) \\ + \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t - \gamma_1(t)), \ldots, \mathbf{x}(t - \gamma_k(t))), \\ \mathbf{x}_{t_0} = \boldsymbol{\phi}, \end{aligned}\right\} \tag{41}$$

exists and it is unique on $[t_0, T)$; moreover,

$$\sup_{t_0 \leqslant t \leqslant T} |\mathbf{x}(t_0, \boldsymbol{\phi})(t) - \mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi})(t)| \to 0, \quad l \to +\infty, \tag{42}$$

where $\mathbf{x}^{(l,0)}:[t_0, T) \to R^n$ is the 0th vector component of the solution $(\mathbf{x}^{(l,0)}, \ldots, \mathbf{x}^{(l,l)})^T$ of the initial value problem

$$\left.\begin{aligned} \dot{\mathbf{x}}^{(l,0)}(t) &= A_0(t)\mathbf{x}^{(l,0)}(t) + \sum_{j=1}^{l}\left[Q\left(t, -(j-1)\frac{\gamma(t)}{l}\right) - Q\left(t, -j\frac{\gamma(t)}{l}\right)\right] \\ &\quad \times \mathbf{x}^{(l,j)}(t) + g(t, \mathbf{x}^{(l,i_1(t))}(t), \ldots, \mathbf{x}^{(l,i_k(t))}(t)), \\ \dot{\mathbf{x}}^{(l,i)}(t) &= -\left(1 - i\frac{\dot{\gamma}(t)}{l}\right)\frac{l}{\gamma(t)}[\mathbf{x}^{(l,i)}(t) - \mathbf{x}^{(l,i-1)}(t)], \quad i = \overline{1, l}, \end{aligned}\right\} \tag{43}$$

$$\mathbf{x}^{(l,i)}(t_0) = \boldsymbol{\phi}\left(-i\frac{\gamma(t_0)}{l}\right), \quad i = \overline{0, l} \tag{44}$$

where

$$i_j(t) = \left[l\frac{\gamma_j(t)}{\gamma(t)}\right] := \text{integer part of } l\frac{\gamma_j(t)}{\gamma(t)}, \quad t_0 \leqslant t \leqslant T, \quad j = \overline{1, k}; \tag{45}$$

(b) for any solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ which is twice continuously differentiable on $[t_0 - \gamma(t_0), T]$ there exists a constant $C = C(\mathbf{x}(t_0, \boldsymbol{\phi}))$ depending on $\mathbf{x}(t_0, \boldsymbol{\phi})$ such that

$$|\mathbf{x}(t_0, \boldsymbol{\phi})(t) - \mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi})(t)| \leqslant C(\mathbf{x}(t_0, \boldsymbol{\phi}))\frac{\gamma_1}{l}, \quad t_0 \leqslant t \leqslant T_1, \quad l \geqslant 2, \tag{46}$$

where

$$\gamma_1 = \sup_{t_0 \leqslant t \leqslant T_1} \gamma(t).$$

Proof. The existence and uniqueness of any solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ of equations (41) on $[t_0, T]$ is a corollary of Proposition 2.1.

Now let us define $\bar{V}$ as the set of solutions $\mathbf{x}(t_0, \boldsymbol{\phi})$ when $\boldsymbol{\phi}$ ranges over C_{t_0} and V denotes the set of continuously differentiable solutions of equations (41) on $[t_0 - \gamma(t_0), T]$.

The function $\mathbf{a}^{(l,i)}:[t_0, T] \times \bar{V} \to R^n$, $(i = \overline{0, l})$, is defined by

$$\mathbf{a}^{(l,i)}(t, \mathbf{x}) = \mathbf{x}\left(t \cdot \frac{i}{l}\gamma(t)\right),$$

for any $(t, \mathbf{x}) \in [t_0, T] \times \bar{V}$, and let $\boldsymbol{\alpha}^{(l,i)} = \mathbf{a}^{(l,i)}(t_0, \mathbf{x})$, $\mathbf{x} \in \bar{V}$.

Then, by Proposition 2.3, we have that these V, $\bar{V}$ and $\mathbf{a}^{(l)} = (\mathbf{a}^{(l,0)}, \ldots, \mathbf{a}^{(l,l)})^T$, $\boldsymbol{\alpha}^{(l)} = (\boldsymbol{\alpha}^{(l,0)}, \ldots, \boldsymbol{\alpha}^{(l,l)})^T$ satisfy condition (i) of Proposition 3.1.

Now, we consider the following matrices:

$$A^{(l,0)}(t) = A_0(t), \quad A^{(l,j)}(t) = Q\left(t, -(j-1)\frac{\gamma(t)}{l}\right) - Q\left(t, -j\frac{\gamma(t)}{l}\right), \quad j = \overline{1, l},$$

and

$$D^{(l,i)}(t) = \left(1 - i\frac{\dot{\gamma}(t)}{l}\right)\frac{l}{\gamma(t)}I_n, \quad i = \overline{0, l}.$$

These matrices are continuous, $D^{(l,i)}$, $(i = \overline{0, l})$ is non-negative and diagonal, moreover

$$\begin{aligned} &\mu_0\left[A^{(l,0)}(t) + \sum_{j=1}^{l} \operatorname{abs}(A^{(l,j)}(t))\right] \\ &\quad = \mu_0\left[A_0(t) + \sum_{j=1}^{l} \operatorname{abs}\left(Q\left(t - (j-1)\frac{\gamma(t)}{l}\right) - Q\left(t, -j\frac{\gamma(t)}{l}\right)\right)\right] \leq \lambda_0, \end{aligned} \tag{47}$$

where

$$\lambda_0 = \max_{t_0 \leqslant t \leqslant T} \{\mu_0[A_0(t)] + m_1(t)\},$$

where $m_1(t)$ is given in expression (39).

This means that these matrices satisfy condition (ii) of Proposition 3.1.

Let us define $\mathbf{H}^{(l,i)}:[t_0, T] \times R^{N_l} \to R^n$ by

$$\mathbf{H}^{(l,0)}(t, \mathbf{x}) = \mathbf{f}(t, \mathbf{x}^{(l,i_1(t))}, \ldots, \mathbf{x}^{(l,i_l(t))}), \quad \mathbf{H}^{(l,1)}(t, \mathbf{x}) = 0,$$

for any $\mathbf{x} = (\mathbf{x}^{(0)}, \ldots \mathbf{x}^{(l)})^{\mathrm{T}} \in R^{N_l}$, where $i_j(t)$ is defined by equation (45).

Then these functions satisfy condition (30) and the initial value problem (43), (44) and (33) are equivalent when $\delta = 0$. Consider a solution $\mathbf{x} = \mathbf{x}(t_0, \boldsymbol{\phi})$ of equation (41) for which $\mathbf{x}(t_0, \boldsymbol{\phi}) \in V$. It is very easy to check that the function

$$\mathbf{a}^{(l)}(t, \mathbf{x}) = \left(\mathbf{x}(t), \mathbf{x}\left(t - \frac{\gamma(t)}{l}\right), \ldots, \mathbf{x}(t - \gamma(t))\right)^{\mathrm{T}}$$

is solution of equation (31) when $\epsilon^{(l)}(\mathbf{x}) = (\epsilon^{(l,0)}(\mathbf{x}), \ldots, \epsilon^{(l,l)}(\mathbf{x}))^{\mathrm{T}}$ is defined by

$$\begin{aligned}\epsilon^{(l,0)}(\mathbf{x})(t) = \sum_{j=1}^{l} \int_{-j(\gamma(t)/l)}^{-(j-1)(\gamma(t)/l)} [\mathrm{d}sQ(t,s)]\left(\mathbf{x}_t\left(-j\frac{\gamma(t)}{l}\right) - \mathbf{x}_t(s)\right) \\ +\mathbf{f}\left(t, x(t), \mathbf{x}\left(t - i_1(t)\frac{\gamma(t)}{l}\right), \ldots, \mathbf{x}\left(t - i_k(t)\frac{\gamma(t)}{l}\right)\right) \\ -\mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t - \gamma_1(t)), \ldots, \mathbf{x}(t - \gamma_k(t)))\end{aligned} \tag{48}$$

and

$$\begin{aligned}\epsilon^{(l,i)}(\mathbf{x})(t) = \left(1 - i\frac{\dot\gamma(t)}{l}\right)\left[\dot{\mathbf{x}}\left(t - i\frac{\gamma(t)}{l}\right) - \left(\frac{\gamma(t)}{l}\right)^{-1}\left(\mathbf{x}\left(t - (i-1)\frac{\gamma(t)}{l}\right)\right.\right. \\ \left.\left. -\mathbf{x}\left(t - i\frac{\gamma(t)}{l}\right)\right)\right], \quad (i = \overline{1,l}, \quad t_0 \leqslant t \leqslant T).\end{aligned} \tag{49}$$

Using inequalities (38) and (39), we have

$$|\epsilon^{(l,0)}(\mathbf{x})(t)|_0 \leqslant (m(t) + m_1(t))\frac{\gamma(t)}{l} \,\|\dot{\mathbf{x}}_t\|_{[-\gamma(t),0]} \tag{50}$$

and

$$|\epsilon^{(l,i)}(\mathbf{x})(t)|_0 \leqslant \max_{-(\gamma(t)/l) \leqslant |s_1 - s_2| \leqslant 0} |\dot{\mathbf{x}}_t(s_2) - \dot{\mathbf{x}}_t(s_1)|_0 \quad \text{on} \quad [t_0, T]. \tag{51}$$

Thus, all conditions of Proposition 3.1, are satisfied for $\delta = 0$ and from inequality (32), we have

$$|\mathbf{x}(t_0, \phi)(t) - \mathbf{x}^{(l,0)}(t_0, \phi)(t)|_0 \leqslant \exp(\lambda_0 t)\int_{t_0}^{t} \exp(-\lambda_0 s)|\epsilon^{(l)}(\mathbf{x})(s)|_0 \,\mathrm{d}s \cdot \exp\left[\int_{t_0}^{t} m(s)\,\mathrm{d}s\right], \tag{52}$$

for any continuously differentiable solution $\mathbf{x} = \mathbf{x}(t_0, \boldsymbol{\phi})$ of equations (41). However, from inequalities (50) and (51), for any $\mathbf{x} = \mathbf{x}(t_0, \boldsymbol{\phi}) \in V$,

$$\max_{t_0 \leqslant t \leqslant T} |\epsilon^{(l)}(\mathbf{x})(t)| \to 0, \quad \text{as} \quad l \to +\infty,$$

thus, condition (42) is valid for any $\mathbf{x}(t_0, \boldsymbol{\phi}) \in \bar{V}$.

To prove inequality (46) we consider a solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ of equation (41) which is twice continuously differentiable on $[t_0 - \gamma(t_0), T]$. Then inequalities (50) and (51) yield

$$|\epsilon^{(l)}(\mathbf{x})(t)|_0 \leqslant \frac{\gamma_1}{l} \max_{t_0 - \gamma(t_0) \leqslant t \leqslant T} |\ddot{x}(t_0, \phi)(t)|_0$$

which, combined with inequality (52), gives inequality (46). The proof is complete.

Remark 3.1

This and following theorems are essential generalizations of a Repin's theorem [1] and contain the approximation result which was given for linear autonomous retarded differential systems in Ref. [2]. Their relation to the results of Banks [3], Banks and Kappel [5] is the following: our two theorems are more general than the last quoted results if we use 0th order splines, but the present theorem does not give any statement for higher order splines. The idea of the proof is different from that used in the mentioned results and it makes it possible to construct a quite new approximation theorem (Theorem 3.4) for a new approximation technique.

Now, we consider the functional differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}_t), \tag{53}$$

on $[t_0, T]$ and we give two approximation theorems.

Theorem 3.2

We suppose that $-\infty < t_0 < T < \infty$ and following assumptions hold:

(i) $\gamma : [t_0, T] \to R_+$ is continuously differentiable, $0 < \gamma(t) < \gamma_1$ and $1 - \dot{\gamma}(t) \geqslant 0$;

(ii) $F(t, \cdot) : C_t \to R^n$, $(t_0 \leqslant t \leqslant T)$, is such that for any $\mathbf{x} \in C([t_0 - \gamma(t_0), T], R^n)$ the function $\mathbf{F}(t, \mathbf{x}_t)$ is continuous on $[t_0, T]$, where $C_t = C([-\gamma(t), 0], R^n)$;

(iii) for any $(t, \mathbf{x})$, $(t, \mathbf{y}) \in [t_0, T) x C([t_0 - \gamma(t_0), T], R^n)$

$$|\mathbf{F}(t, \mathbf{y}_t) - F(t, \mathbf{x}_t)|_0 \leqslant m \, \|\mathbf{y}_t - \mathbf{x}_t\|_{[-\gamma(t), 0]},$$

where m is a non-negative constant;

(iv) there exists a function sequence $\mathbf{H}^{(l,0)} : [t_0, T] \times R^{N_l} \longrightarrow R^n$, $(N_l = (l+1)n,\ l \geqslant 2)$, such that $\mathbf{H}^{(l,0)}$ is continuous and for any continuously differentiable function $\mathbf{x} : [t_0 - \gamma(t_0), T] \to R^n$,

$$|\mathbf{F}(t, \mathbf{x}_t) - \mathbf{H}^{(l,0)}(t, \mathbf{a}^{(l)}(t, \mathbf{x}))|_0 \leqslant m \frac{\gamma_1}{l} \|\dot{x}_t\|_{[-\gamma(t), 0]} \quad t_0 \leqslant t \leqslant T,$$

where

$$\mathbf{a}^{(l)}(t, \mathbf{x}) = \left(\mathbf{x}(t), \mathbf{x}\left(t - \frac{\gamma(t)}{l}\right), \ldots, \mathbf{x}(t - \gamma(t))\right), \quad (l \geqslant 2),$$

moreover

$$|\mathbf{H}^{(l,0)}(t, \mathbf{x}) - \mathbf{H}^{(l,0)}(t, \mathbf{y})|_0 \leqslant m |\mathbf{x} - \mathbf{y}|_0, \quad (t, \mathbf{x}), (t, \mathbf{y}) \in [t_0, T) \times R^{N_l},$$

for any $l \geqslant 2$.

Then for any $\boldsymbol{\phi} \in C_{t_0}$ equation (53) has exactly one solution $\mathbf{x}(t_0, \boldsymbol{\phi})$ on $[t_0 - \gamma(t_0), T]$ belonging to initial point $(t_0, \boldsymbol{\phi})$ and

$$\max_{t_0 \leqslant t \leqslant T} |\mathbf{x}(t_0, \boldsymbol{\phi})(t) - \mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi})(t)|_0 \to 0, \quad l \to +\infty;$$

moreover, if $\mathbf{x}(t_0, \phi)$ is twice continuously differentiable on $[t_0 - \gamma(t_0), T]$ then

$$|\mathbf{x}(t_0, \boldsymbol{\phi})(t) - \mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi})(t)| \leqslant C \frac{\gamma_1}{l}, \quad l \geqslant 2, \quad t_0 \leqslant t \leqslant T, \tag{54}$$

where C is suitable constant, and $\mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi}):[t_0-\gamma(t_0), T]\to R^n$ is the 0th vector component of the solution $\mathbf{x}^{(l)}(t_0, \boldsymbol{\phi}) = (\mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi}), \ldots, \mathbf{x}^{(l,l)}(t_0, \boldsymbol{\phi}))^{\mathrm{T}}$ of

$$\left.\begin{aligned}\dot{\mathbf{x}}^{(l,0)}(t) &= \mathbf{H}^{(l,0)}(t, \mathbf{x}^{(l)}(t)), \quad t_0 \leqslant t \leqslant T,\\ \dot{\mathbf{x}}^{(l,i)}(t) &= -\left(1 - i\frac{\dot{\gamma}(t)}{l}\right)\frac{l}{\gamma(t)}[\mathbf{x}^{(l,i)}(t) - \mathbf{x}^{(l,i-1)}(t)], \quad t_0 \leqslant t \leqslant T, \quad l \geqslant 2,\end{aligned}\right\} \tag{55}$$

and

$$\mathbf{x}^{(l,i)}(t_0) = \boldsymbol{\phi}\left(-i\frac{\gamma(t_0)}{l}\right), \quad i = \overline{0, l}. \tag{56}$$

Proof. The proof is essentially similar to that of Theorem 3.1, therefore we do not detail it. For the approximation of the Carathéodory solutions we give the following theorem.

Theorem 3.3

We suppose that $\infty < t_0 < T < \infty$ and

(i) $\gamma:[t_0, T]\to R_+$ is given as in condition (i) of Theorem 3.2;

(ii) $\mathbf{F}(t,\cdot):C_t \to R^n$, $(t_0 \leqslant t \leqslant T)$, is such that for any $\mathbf{x}\in C([t_0-\gamma(t_0), T], R^n)$, the function $\mathbf{F}(t, \mathbf{x}_t)$ is continuous and for any $\mathbf{x}\in C([t_0-\gamma(t_0), T], R^n)$ it is integrable on $[t_0, T]$;

(iii) for any $(t, \mathbf{x}), (t, \mathbf{y}) \in [t_0, T]\times C([t_0-\gamma(t_0), T], R^n)$,

$$|\mathbf{F}(t, \mathbf{x}_t) - F(t, \mathbf{y}_t)|_0 \leqslant m\left\{\sum_{i=0}^{N} |\mathbf{x}_t(-\gamma_i(t)) - \mathbf{y}_t(-\gamma_i(t))|_0 + \int_{-\gamma(t)}^{0} |\mathbf{x}_t(s) - \mathbf{y}_t(s)|_0\,\mathrm{d}s\right\}, \tag{57}$$

where $m \geqslant 0$ is a constant, $\gamma_i(t)$ is differentiable on $[t_0, T]$ and $\dot{\gamma}_i(t) \leqslant 1-\epsilon$, $(i = \overline{0, N}$, $t_0 \leqslant t \leqslant T)$, where $\epsilon \in (0, 1)$;

(iv) there exists a function sequence $\mathbf{H}^{(l,0)}:[t_0, T]\times R^{N_l}\to R^n$, $l \geqslant 2$, such that $\mathbf{H}^{(l,0)}$ satisfies condition (iv) of Theorem 3.2. Then for any $\boldsymbol{\phi}\in L_{t_0}$ the equation (53) has exactly one Carathéodory solution on $[t_0-\gamma(t_0), T]$ and the statements of Theorem 3.2 are valid, where now $\mathbf{x}^{(l)}(t_0, \boldsymbol{\phi}) = (\mathbf{x}^{(l,0)}(t_0, \boldsymbol{\phi}), \ldots, \mathbf{x}^{(l,l)}(t_0, \boldsymbol{\phi}))$ denotes the solution of equation (55) with initial conditions.

$$\mathbf{x}^{(l,0)}(t_0) = \boldsymbol{\phi}(0), \quad \mathbf{x}^{(l,i)}(t_0) = \frac{l}{\gamma(t_0)}\int_{-i[\gamma(t_0)/l]}^{-(i-1)[\gamma(t_0)/l]} \boldsymbol{\phi}(s)\,\mathrm{d}s, \quad i = \overline{1, l}. \tag{58}$$

Proof. The proof is similar to that of Theorem 3.1, therefore we only indicate the major changes. First we define V as the set of continuously differentiable solutions of equation (53) on $[t_0-(t_0), T]$ and $\bar{V}$ as the set of Carathéodory solutions. Then using Proposition 2.4 we have that $\mathbf{a}^{(l)}$ and $\boldsymbol{\alpha}^{(l)} = (\boldsymbol{\alpha}^{(l,0)}, \ldots, \boldsymbol{\alpha}^{(l,l)})^{\mathrm{T}}$ satisfy condition (i) of Proposition (3.1), where now

$$\boldsymbol{\alpha}^{(l,0)}(\mathbf{x}) = \mathbf{x}(t_0), \quad \boldsymbol{\alpha}^{(l,i)}(\mathbf{x}) = \frac{l}{\gamma(t_0)}\int_{-i[\gamma(t_0)/l]}^{-(i-1)[\gamma(t_0)/l]} |\mathbf{x}_{t_0}(s)|\,\mathrm{d}s, \quad i = \overline{1, l}, \quad \text{for any } \mathbf{x}\in\bar{V}.$$

To complete the proof, it only remains to follow the rest of the proof of Theorem 3.1 with the necessary changes.

Example 3.1

The equation

$$\dot{x}(t) = -g(t, x(t)) + g(t-\gamma, x(t-\gamma)), \quad t \geqslant 0, \tag{59}$$

is important in the literature as a model of infection or a compartmental system with pipe (see for example Refs [11, 12, 19], where $\gamma > 0$ is constant and g is non-negative Lipschitz-continuous function.

Using the approximation procedure, from Theorems 3.2 and 3.3 we have the following approximation equations:

$$\left.\begin{aligned} \dot{x}^{(l,0)}(t) &= -g(t, x^{(l,0)}(t)) + g(t, x^{(l,l)}(t)), \\ \dot{x}^{(l,i)}(t) &= -\frac{l}{\gamma} x^{(l,i)}(t) + \frac{l}{\gamma} x^{(l,i-1)}(t), \quad i = \overline{1, l}, \end{aligned}\right\} \tag{60}$$

with the initial conditions

$$x^{(l,i)}(0) = x\left(-i\frac{\gamma}{l}\right), \quad i = \overline{0, l}, \tag{61}$$

and

$$x^{(l,0)} = x(0), \quad x^{(l,i)}(0) = \frac{l}{\gamma} \int_{-i[\gamma/l]}^{-(i-1)[\gamma/l]} x(s)\,\mathrm{d}s, \quad i = \overline{1, l}, \tag{62}$$

for a continuous and a Carathéodory solution of equation (59), respectively.

Now, we investigate the physical meaning of equation (59) to get new approximation equations. Let us start from a model for some infections diseases given by Cooke and Kaplan [19] and Cooke and Yorke [20].

Here, we do not write down all assumptions of this S–I–S model, only the following out of them.

Population has constant size N, that is it is fixed, isolated and the population is divided into two disjoint classes: susceptibles and infectives.

The numbers of susceptibles and infectives are denoted by $N \cdot S(t)$ and $N \cdot I(t)$, respectively, so that

$$S(t) + I(t) = 1, \quad S(t), \quad I(t) \geqslant 0.$$

The number of new infection is represented by $N \cdot f(t, I(t))$ at time t and the period of time an individual is infective is a fixed positive constant τ.

Then the model equation is

$$\dot{I}(t) = f(t, I(t)) - f(t-\tau, I(t-\tau))$$

for I and

$$\dot{S}(t) = -g(t, S(t)) + g(t-\tau, S(t-\tau)), \tag{63}$$

for S, where $g(t, u) = f(t, 1-u)$.

Equation (63) and its generalized form

$$\dot{x}(t) = -g(t, x(t)) + \int_0^{\gamma} f(t-u, x(t-u))\,\mathrm{d}F(u), \quad t \geqslant 0, \tag{64}$$

have also important role in compartmental models with pipes [11] and some general population growth models [21]. The initial condition for this equation is given by

$$x_0(s) = x(s) = \phi(s), \quad -\gamma \leqslant s \leqslant 0, \quad \phi \in C([-\gamma, 0], R). \tag{65}$$

After this we consider that case when equation (64) describes a population growth model: $x(t)$ denotes the population size at time t, $g(t, x)$ is the death rate, being a function of t and population size. The function $f(t, x)$ is the maternity rate of egg-laying, depending on time t and population size x. The function $F: [0, \gamma] \to [0, 1]$ is probability distribution function representing the distribution of gestation time, and γ denotes the maximal gestation time. Thus, the integral in equation (64) gives the rate of appearance of new individuals at time t due to eggs laid at all previous times and the integral

$$\int_0^{\gamma} \left(\int_{t-u}^{t} f(s, x(s))\,\mathrm{d}s \right) \mathrm{d}F(u),$$

represents the number of new individuals in the period $[t-\gamma, t]$. Equation (64) has the following equivalent integrated form:

$$x(t)+\int_0^{\gamma}\left(\int_{t-u}^{t} f(s, x(s))\,\mathrm{d}s\right)\mathrm{d}F(u) = -\int_0^t f_0(s, x(s))\,\mathrm{d}s + c(\phi), \quad t \geqslant 0, \tag{66}$$

where $f_0(t, x) = g(t, x) - f(t, x)$ and

$$c(\phi) = \phi(0) + \int_0^{\gamma}\left(\int_{-u}^{0} f(s, \phi(s))\,\mathrm{d}s\right)\mathrm{d}F(u). \tag{67}$$

This form represents "the material conservation law", and sometimes it is important for the determination of the steady state of equation (64). Our aim is to derive a scheme, which gives simultaneously, a good approximation of delay differential equation (64) and its equivalent integral equation (66), too.

Suppose that $F(u)$ is continuous at $u = 0$ and divide the interval $(0, \gamma)$ into $l(\geqslant 2)$ subintervals of length γ/l. Then for any $x \in C([-\gamma, \infty), R)$, we have

$$\int_0^{\gamma} f(t-u, x(t-u))\,\mathrm{d}F(u) = \sum_{j=1}^{l} \delta^{(l,j)} f\left(t - j\frac{\gamma}{l}, x\left(t - j\frac{\gamma}{l}\right)\right) + \epsilon^{(l,0)}(\mathbf{x})(t),$$

where

$$\delta^{(l,j)} = F\left(j\frac{\gamma}{l}\right) - F\left((j-1)\frac{\gamma}{l}\right), \quad \sum_{j=1}^{l} \delta^{(l,j)} = 1$$

and $\epsilon^{(l,0)}(\mathbf{x})(t)$ is a continuous function on $[0, \infty)$ such that

$$\max_{0 \leqslant t \leqslant T} |\epsilon^{(l,0)}(x)(t)| \to 0, \quad \text{as } l \to \infty, \quad \text{for any } T \in [0, \infty). \tag{68}$$

Moreover,

$$\int_0^{\gamma}\left(\int_{t-u}^{t} f(s, x(s))\,\mathrm{d}s\right)\mathrm{d}F(u) = \int_0^{\gamma}\left(\int_0^{v} f(t-\tau,\ x(t-\tau)\,\mathrm{d}\tau)\,\mathrm{d}F(v)\right)$$

$$= \sum_{j=1}^{l} \delta^{(l,j)} \sum_{i=1}^{j} \frac{\gamma}{l} f\left(t - i\frac{\gamma}{l},\ x\left(t - i\frac{\gamma}{l}\right)\right) + \eta_2^{(l)}(\mathbf{x})(t),$$

that is,

$$\int_0^{\gamma}\left(\int_{t-u}^{t} f(s, x(s))\,\mathrm{d}s\right)\mathrm{d}F(u) = \sum_{i=1}^{l}\left(\sum_{j=1}^{l} \delta^{(l,j)} \frac{\gamma}{l} f\left(t - i\frac{\gamma}{l}, x\left(t - i\frac{\gamma}{l}\right)\right)\right) + \eta_2^{(l)}(x)(t), \tag{69}$$

where $\eta_2^{(l)}(x)(t)$ is a continuous function on $[0, \infty)$ such that

$$\max_{0 \leqslant t \leqslant T} \eta_2^{(l)}(x)(t) \to 0, \quad \text{as } l \to +\infty, \quad \text{for any } T \in [0, \infty). \tag{70}$$

Now, for any $x \in C([-\gamma, \infty), R)$ put

$$a^{(l,0)}(t, x) = x(t), \quad a^{(l,i)}(t, x) = \frac{\gamma}{l}\sum_{j=i}^{l} \delta^{(l,j)} \cdot f\left(t - i\frac{\gamma}{l},\ x\left(t - i\frac{\gamma}{l}\right)\right), \quad i = 1, l.$$

Let us suppose that $f(t, x)$ is a continuously differentiable function on $[-\gamma, \infty) \times R$ and $x = x(t_0, \phi)(\phi \in C([-\gamma, 0], R))$, is a continuously differentiable solution of equations (64) and (65).

Then, for this x

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}a^{(l,0)}(t,x) &= -f_0(t,a^{(l,0)}(t,x)) + \sum_{j=1}^{l}\frac{l}{\gamma}\left(\delta^{(l,j)}\Big/\sum_{k=j}^{l}\delta^{(l,k)}\right), \quad a^{(l,j)}(t,x)+\epsilon^{(l,0)}(x)(t),\\
\frac{\mathrm{d}}{\mathrm{d}t}a^{(l,1)}(t,x) &= -\frac{l}{\gamma}a^{(l,1)}(t,x)+f(t,a^{(l,0)}(t,x)+\epsilon^{(l,1)}(x)(t)\\
\frac{\mathrm{d}}{\mathrm{d}t}a^{(l,i)}(t,x) &= -\frac{l}{\gamma}a^{(l,i)}(t,x)+\frac{l}{\gamma}\frac{\sum_{j=i}^{l}\delta^{(l,j)}}{\sum_{j=i-1}^{l}\delta^{(l,j)}}a^{(l,i-1)}(t,x)+\epsilon^{(l,i)}(x)(t), \quad (i=\overline{2,l},\ t\geqslant 0),
\end{aligned}\right\} \tag{71}$$

where

$$\epsilon^{(l,i)}(\mathbf{x})(t)=\frac{\gamma}{l}\sum_{j=1}^{l}\delta^{(l,j)}\left[\frac{\mathrm{d}}{\mathrm{d}t}f\left(t-i\frac{\gamma}{l},x\left(t-i\frac{\gamma}{l}\right)\right)\right.$$

$$\left.-\frac{f\left(t-(i-1)\frac{\gamma}{l},x\left(t-(i-1)\frac{\gamma}{l}\right)\right)-f\left(t-i\frac{\gamma}{l},x\left(t-i\frac{\gamma}{l}\right)\right)}{\frac{\gamma}{l}}\right], \quad i=\overline{1,l}.$$

Thus, for $\epsilon^{(l)}(x)(t)=(\epsilon^{(l,0)}(x)(t),\ldots,\epsilon^{(l,l)}(x)(t))^{\mathrm{T}}$ we have

$$\begin{aligned}
|\epsilon^{(l)}(x)(t)|_1 &= \epsilon^{(l,0)}(x)(t)+\sum_{i=1}^{l}\epsilon^{(l,i)}(x)(t)\leqslant\epsilon^{(l,0)}(x)(t)\\
&\quad+\left(\sum_{j=1}^{l}\delta^{(l,j)}\right)\max_{\substack{0\leqslant s_1,s_2\leqslant T\\ |s_2-s_1|\leqslant\gamma/l}}|f'(s_2,x(s_2))-f'(s_1,x(s_1))|\\
&\leqslant C_1\max_{0\leqslant s_1,s_2\leqslant T}|f'(s_2,x(s_2))-f'(s_1,x(s_1))|\to 0, \quad \text{as } l\to+\infty, \quad |s_2-s_1|\leqslant\frac{\gamma}{l}
\end{aligned}$$

for any fixed $T\in[0,\infty)$, where

$$C_1=1+F(\gamma)-F(0),$$

and

$$f'(t,x(t))=\frac{\mathrm{d}}{\mathrm{d}t}f(t,x(t)).$$

From equations (71), ignoring the terms $\epsilon^{(l)}(x)(t)$ which tend to 0 if $l\to+\infty$, we get the following approximating ordinary differential equations of equation (64):

$$\left.\begin{aligned}
\dot{x}^{(l,0)}(t) &= -g(t,x^{(l,0)}(t))+f(t,x^{(l,0)}(t))+\sum_{j=1}^{l}\frac{l}{\gamma}\left(\delta^{(l,j)}\Big/\sum_{k=j}^{l}\delta^{(l,k)}\right)x^{(l,j)}(t),\\
\dot{x}^{(l,1)}(t) &= -\frac{l}{\gamma}x^{(l,1)}(t)+f(t,x^{(l,0)}(t)),\\
\dot{x}^{(l,i)}(t) &= -\frac{l}{\gamma}x^{(l,i)}(t)+\frac{l}{\gamma}\left(\sum_{j=1}^{l}\delta^{(l,j)}\Big/\sum_{j=i-1}^{l}\delta^{(l,j)}\right)x^{(l,i-1)}(t) \quad (t\geqslant 0,\quad i=\overline{2,l},\quad l\geqslant 2),
\end{aligned}\right\} \tag{72}$$

with the initial conditions

$$x^{(l,0)}(0)=\phi(0), \quad x^{(l,i)}(0)=\int_{(i-1)[\gamma/l]}^{i[\gamma/l]}\left(\int_{-u}^{0}f(s,\phi(s)\,\mathrm{d}s)\,\mathrm{d}F(u)\right), \quad i=\overline{1,l}. \tag{73}$$

Now let us define

$$A_1^{(l)} = (a_{ij}^{(l)}) \quad (0 \leqslant i, j \leqslant l) \quad \text{by} \quad a_{ii}^{(l)} = -\frac{l}{\gamma}, \quad a_{i,i-1} = \frac{l}{\gamma}\left(\sum_{j=1}^{l} \delta^{(l,j)} \Big/ \sum_{j=i-1}^{l} \delta^{(l,j)}\right),$$

$$a_{0i} = \frac{l}{\gamma}\left(\delta^{(l,i)} \Big/ \sum_{k=i}^{l} \delta^{(l,k)}\right) \quad \text{for} \quad 1 \leqslant i \leqslant l$$

and

$$a_{ij} = 0 \quad \text{when} \quad j \neq i \quad \text{and} \quad j \neq i-1, \quad 0 \leqslant i, \quad j \leqslant l.$$

Then $\mu_1[A_1^{(l)}] = 0$, $(l \geqslant 2)$, and thus all conditions are satisfied to use Proposition 3.1 for this new approximation scheme. Consequently, we may state the following new approximation result.

Theorem 3.4

Suppose that $\gamma > 0$ is a real number, $F:[0, \gamma] \to [0, 1]$ is probability distribution function such that F is continuous at $t = 0$, furthermore the functions $f:[-\gamma, \infty) \times R \to R$ and $g:[0, \infty) \times R \to R$ are continuously differentiable.

Then for any $\phi \in C([-\gamma, 0], R)$ the solution $x(0, \phi)$ of the initial value problem (64) and (65) exists, it is unique on $[0, \infty)$ and

$$\max_{0 \leqslant t \leqslant T} |x(0, \phi)(t) - x^{(l,0)}(0, \phi)(t)| \to 0, \quad \text{as } l \to +\infty, \tag{74}$$

for any $T \in [0, \infty)$, where $x^{(l,0)}(0, \phi)$ is the 0th component of the solution $x^{(l)}(0, \phi) = (x^{(l,0)}(0, \phi), \ldots, x^{(l,l)}(0, \phi))^{\mathrm{T}}$ of the initial value problem (72) and (73).

Moreover, for $x^{(l)}(0, \phi)$, $(l \geqslant 2)$, we have

$$\sum_{i=0}^{l} x^{(l,i)}(0, \phi)(t) = -\int_{t_0}^{t} [g(s, x^{(l,0)}(s)) - f(s, x^{(l,0)}(s))]\, ds + c(\phi), \quad t \geqslant 0, \tag{75}$$

where $c(\phi)$ is defined by equation (67).

To show what is the use of relation (75) we consider the simpler equation

$$x(t) = -g(x(t)) + g(x(t-\gamma)), \quad t \geqslant 0, \tag{76}$$

where $g: R \to R$ is a continuously differentiable and strictly monotone decreasing function. Then, it is known from Ref. [22] that for any solution $x(0, \phi)(t)$, $(\phi \in C([-\gamma, 0], R))$, the limit

$$d(\phi) = \lim_{t \to +\infty} x(0, \phi)(t)$$

exists, but we cannot determine $d(\phi)$ directly form equation (76). At the same time from the material conservation law

$$\dot{x}(t) + \int_{t-\gamma}^{t} g(x(s))\, ds = \phi(0) + \int_{-\gamma}^{0} g(\phi(s))\, ds,$$

we have an equation

$$d + \gamma g(d) = \phi(0) + \int_{-\gamma}^{0} g(\phi(s))\, ds = : c(\phi),$$

which has unique solution $d = d(\phi)$ for any fixed ϕ.

The first approximation method yields the equations

$$\dot{x}^{(l,0)}(t) = -g(x^{(l,0)}(t)) + g(x^{(l,l)}(t))$$

$$\dot{x}^{(l,i)}(t) = -\frac{l}{\gamma} x^{(l,i)}(t) + \frac{l}{\gamma} x^{(l,i-1)}(t), \quad i = \overline{1, l}$$

with initial values

$$x^{(l,i)}(0) = \phi\left(-i\frac{\gamma}{l}\right), \quad i = \overline{0, l},$$

which does not give useful information on $d(\phi)$.

Applying the second approximation method we get the following system:

$$\begin{aligned} \dot{x}^{(l,0)}(t) &= -g(x^{(l,0)}(t)) + \frac{l}{\gamma} x^{(l,l)}(t); \\ \dot{x}^{(l,1)}(t) &= -\frac{l}{\gamma} x^{(l,1)}(t) + g(x^{(l,0)}(t)); \\ \dot{x}^{(l,i)}(t) &= -\frac{l}{\gamma} x^{(l,i)}(t) + \frac{l}{\gamma} x^{(l,i-1)}(t), \quad i = \overline{2, l}, \end{aligned} \tag{77}$$

with initial conditions

$$x^{(l,0)}(0) = \phi(0), \quad x^{(l,i)}(0) = \int_{-i(\gamma/l)}^{-(i-1)(\gamma/l)} g(\phi(s))\,\mathrm{d}s, \quad i = \overline{1, l}$$

and parallel to this we have the next connection

$$\sum_{i=0}^{l} x^{(l,i)}(t) = \phi(0) + \int_{-\gamma}^{0} g(\phi(s))\,\mathrm{d}s, \quad t \geqslant 0. \tag{78}$$

Using a theorem of Ref. [23] we obtain that

$$d^{(l,i)} = \lim_{t \to +\infty} x^{(l,i)}(t)$$

exists. Moreover, from equations (77) and (78), we have

$$d^{(l,l)} = d^{(l,l-1)} = \cdots = d^{(l,1)} = \frac{\gamma}{l} g(d^{(l,0)})$$

and

$$d^{(l,0)} + \gamma g(d^{(l,0)}) = \phi(0) + \int_{-\gamma}^{0} g(\phi(s))\,\mathrm{d}s,$$

that is, $d^{(l,0)} = d(\phi)$, for any $l \geqslant 2$.

Remark 3.2

The last theorem can be extended to systems with delays, for example, to model equations of compartmental system with pipes (see Ref. [24]).

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